Noncommutative Field Theories and Geometry

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Plan of the talks

- Lecture 1: NC field theories: Lorentz symmetry breakdown; IR/UV mixing; Stripes phase; solitons; Unitarity; Fuzzy sphere.

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- hep-th/0508002, 0508151, 0602265, 0604061, 0608138, 0608179 + on going .. Balachandran, Govindarajan, Sachin Vaidya, Giorgio Immirzi, Seckin, Kumar Gupta, Marco Panero, Gianpiero Mangano, Alexander Pinzul, Quereshi....

Motivations.....

◇ Quantum gravity -at Planck length - folklore- must have - noncommutative geometric structure - limit of classical gravity - emerge - commutative geometry of spacetime we know. Just like:

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- Expectation:

$$\lim_{Planck\ length \longrightarrow 0} \quad Non\ commutative\ geometry$$
$$=$$
$$Commutative\ Geometry$$

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- ◇ The above arguments have been posed in two independent places. (1) Sergio Doplicher's paper. (2)Podles lectures on quantum groups - where it is mentioned that Nahm has posed the questions and the need to go beyond conventional ideas of geometries.

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- The above is from “On the hypotheses which lie at the bases of geometry”, Bernhard Riemann, 1854 (from the translation by W K Clifford).

QFT in Moyal spacetimes...

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- This can be understood by the introduction of star product rule in the algebra of functions on R^4. The multiplication map of algebra of functions (*on Moyal plane*) $\mathcal{A}_\theta(R^d)$ is $f \, * \, g \;=\; m_\theta(f \otimes g) \;=\; m_0(F_\theta(f \otimes g))$

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◇ In commutative spacetime we have pointwise multiplication $m_0(F_{\theta=0}(f \otimes g))$.

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- The algebra $\mathcal{A}_\theta(R^d)$ of functions on a noncommutative space can be traced back to quantum mechanics. Many of the techinques of geometry of quantum spaces can be used with advantage. However we are not dealing with inherently quantum mechanical spaces, but only use the techniques to represent classical noncommutative spaces.

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- With the above in mind let us consider a scalar field theory in NC (R^d) space with the Lagrangian (density)

$$\mathcal{L}_* = \frac{1}{2}\partial_\mu\Phi * \partial^\mu\Phi - \frac{1}{2}m^2\Phi * \Phi - \frac{\lambda}{4!}\Phi * \Phi * \Phi * \Phi\,,$$

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- We assume noncommutativity is restricted to space-space coordinates.

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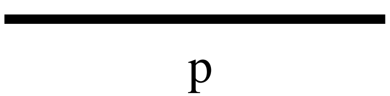

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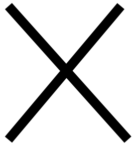

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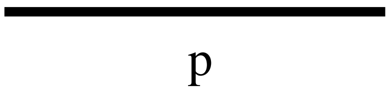

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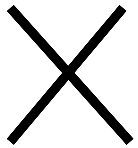

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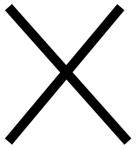

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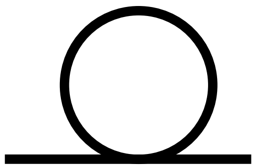
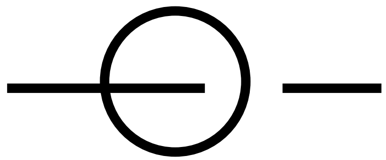


PLANAR



NONPLANAR

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$$I_{\text{nonplanar}} = \frac{\lambda}{(2\pi)^4} \int d^4k \, \frac{1}{k^2 + m^2} \, e^{i \, p \cdot \theta \cdot k}$$

- Planar diagram is quadratically divergent and requires cut-off to make it finite. The nonplanar diagram is finite with an effective cut-off: $\Lambda_{eff}^2 = \frac{1}{\frac{1}{\Lambda^2} + p \cdot \theta^2 \cdot p}$ Ultraviolet divergence is restored in nonplanar diagram either at $\theta \to 0$ or in the infrared limit $p \to 0$. This leads to well known IR/UV mixing.

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- It leads to a new phase for the theory known sometimes as stripe phase or nonuniform phase in addition to order and disorder phases.

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- It is also claimed that Unitarity will be violated (again attributed to IR/UV mixing) in space-time noncommutativity.
- Gauge transformations get modified to take into account new multiplication law.

Gauge theories...

- Conventional Gauge transformations will not close with the new multiplication map given as star product. For this one introduces star gauge transformations: Under star gauge transformation $A_\mu(x) \longrightarrow g(x) * A_\mu(x) * g^\dagger(x) - g(x) * \partial_\mu g(x)^\dagger$.

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- The NC field strength
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu * A_\nu - A_\nu * A_\mu)$
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- Since gauge transformations are introduced in this way there is no way to get gauge groups other than $U(N)$. Infact there is no standard model unless we extend to include $U(1)$.

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- For example the field strength $F_{\mu\nu}$ is expanded as:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ &- \frac{1}{2}\theta^{\rho\gamma}(\partial_\rho A_\mu \partial_\gamma A_\nu - \partial_\rho A_\nu \partial_\gamma A_\mu) + \mathcal{O}(\theta^2) \end{aligned}$$

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- Phenomenological consequences have been worked out.

Solitons in Moyal...

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- The static solitons are obtained by minimising the energy:

$$E = \int d^2x \left[(\partial \phi)^2 + V(\phi) \right]$$

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◇ The solution is given by $\phi = \sum_i \lambda_i P_i$ where λ_i are solutions of $V(\lambda) = 0$ and $P_i^2 = P_i$ are the orthogonal rank-1 projectors. For example the simplest solution will use the projector $P = |\,0\rangle\langle 0\,|$. These solutions are harmonic oscillator wavefunctions whose width is determined by the θ parameter.

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- We have vortex solutions given by $D\phi = 0$ and $V'(\phi) = 0$.

Solitons in 4 dimensions...

- The exact solutions can be obtained by solution generating technique: harvey For example exact soln is:

$$\phi = \lambda(1 - P); \;\; A = \frac{-i}{\theta}\left(\sqrt{\frac{N+1}{N+2}} - 1\right) a^{\dagger}.$$

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- we will combine the kink solution along z axis and nc soliton in the $x-y$ plane. In the $\theta \longrightarrow \infty$ we minimise the energy functional:

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- The solution for $\theta = \infty$ and gauge potential zero is:

$$\phi = a(P\phi_0(z) + iQ), \; P^2 = P, \; Q = 1 - P$$

where $\phi_0(z)$ is the kink solution of 1+1 dimensional theory.

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- The above solution leads to soliton mass characterised by the rank of the projector P. It has the correct behaviour at ∞. One can order by order in $\frac{1}{\theta}$ solve for the solutions for finite and large θ

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- For the case of $P = |0\rangle\langle 0|$ we have

$$S = \sum |n\rangle\langle n+1|$$

Symmetries in NC theories...

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- But there is a way out for preserving Poincare symmetry in NC theories. Also Unitarity issue is more subtle than the above arguments.

Fuzzy torus, sphere,...

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  These appear naturally in certain string theory compactifications, showing possibly the consistancy of these backgrounds.
- The algebra has finite dimensional representations if θ is a root of unity. But for irrational multiples of 2π representations are infinite dimensional. For QFT's on torus one can consider these tori as regularisation.

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- Another interesting algebra is discretisation for S^2 obtained from the condition: $\sum x_i^2 = R^2$. Commutative algebra of functions on S^2 are obtained by homogeneous polynomials of x_i with the above condition. Fuzzy spheres S_F^2 are obtained by:

$$[\, x_i, x_j] = i\theta \, \epsilon_{ijk} x_k$$

and the condition as above.

Fuzzy torus, sphere,...

◇ Using the representation theory of $SU(2)$ one can consider field theory on S^2_F as regularised version of continuum theory. This has the major advantage of consitently having full $SU(2)$ symmetry at the regularised level. In addition it nicely avoids Fermion doubling problem by naturally incorporating Ginsparg-Wilson mechanism.

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- ◇ Fuzzy spheres have nice limits to sphere, plane and Moyal plane. Analytical and numerical studies have been done extensively on these and explicite demonstration of existance of three phases, viz., ordered, disordered and nonuniform phases have been done.
- ◇ In addition QFT's on $S_F^2 \otimes R_1$ exhibit solitons too vaidya.

Fuzzy sphere,...

◇ The action for scalar field on Fuzzy sphere S_F^2 is

$$S(\Phi) = \frac{4\pi}{N} \mathrm{Tr}\left[\Phi\left[L_i, \left[L_i, \Phi\right]\right] + R^2\left(r\Phi^2 + \lambda\Phi^4\right)\right].$$

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- The above is a matrix model and is amenable to simulations easily. The fields on fuzzy spaces are explicitly finite and do not have the IR/UV mixing vaidya,madore. But there is an anomaly in the finite case which reveals itself as generating the IR/UV mixing in the continuum.

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- There is lot of confusion about taking the limit of continuum in these models and it has been pointed out various possibilities do exist.

QFT on Fuzzy sphere-Phase structure..

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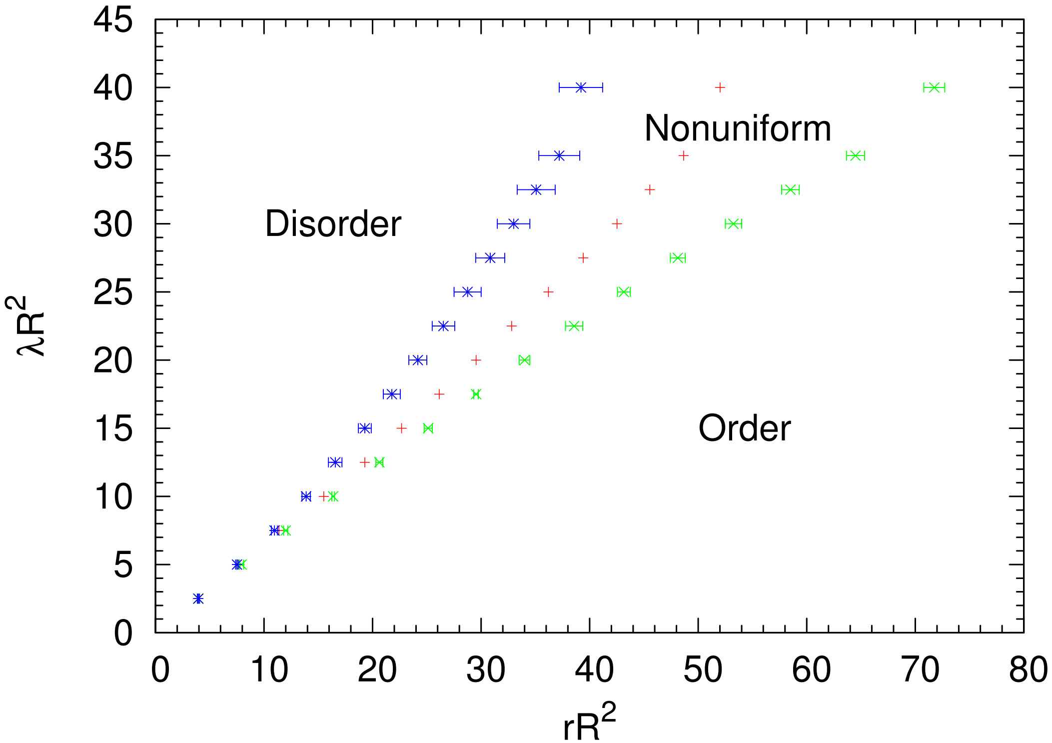
- We find appearance of new phase called nonuniform phase where $\langle Tr\Phi\rangle = 0$.
- The phase diagram is shown below[crdas,digal,trg].



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- These fuzzy spaces have all been put to use for characterising quantum hall droplets as well as higher dimensional QH states nair etal.,.
- CP^1 Sigma models on NC space have been shown to have interesting topological solitons bal,immirzi,trg,hari.